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Thermal Representation of the Energy Density in Bounded Spaces

G. Cocho,[1] S. Hacyan,[2] A. Sarmiento,[2] and F. Soto[1]

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The energy-momentum tensor of a quantum massless free field in a curved spacetime can be written in many cases as an integral with a thermal denominator and a modified phase-space numerator. It is shown that in general the thermal denominator is related to the bounded nature of the system, which in turn implies a representation of the energy density as an infinite numerable sum in Fock space. The modification of the phase-space density is related to the absence of long-wave contributions for nonzero values of the spin.

1. INTRODUCTION

Much research has been carried out on the relation between accelerated or gravitational systems and thermodynamics (Bekenstein, 1973; Hawking, 1975, 1976; Davies, 1978; Sciama *et al.*, 1981; Hacyan *et al.*, 1985; Hacyan, 1985, 1986; Hacyan and Sarmiento, 1986, 1989; Sarmiento *et al.*, 1989). In particular, it has been shown that the energy-momentum tensor of a massless free field of spin s in the presence of an accelerated mirror can be written as (Bunch 1978; Candelas and Deutsch 1977, 1978; Candelas and Dowker, 1979)

$$\langle T_\mu^\nu\rangle = -\frac{h(s)}{32\pi^6}\int_0^\infty \frac{\omega(\omega^2+4\pi^2 s^2 a^2)\,d\omega}{e^{\omega/a}+(-1)^{1+2s}}\,\mathrm{diag}(-1,\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}) \tag{1.1}$$

where a is the local acceleration far from the mirror and $h(s)$ denotes the number of independent helicity states [$h(s)=1$ for $s=0$ and $h(s)=2$ for $s\neq 0$].

A similar formula with a positive sign has been found for the energy-momentum tensor of a massless free field in an Einstein universe of radius

[1]Instituto de Física, Universidad Nacional Autónoma de México, México D.F. 04510, México.
[2]Instituto de Astronomía, Universidad Nacional Autónoma de México, México D.F. 04510, México.

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R and for the neighborhood of a Schwarzschild black hole of mass M (Birrel and Davies, 1982).

These formulas have been derived for spin values $s=0$, 1/2, and 1. Two important features in them are:

1. A *thermal* denominator with a *temperature* related to the parameters of the system [$T=a/2\pi k_B$ for an accelerated system, $T=(2\pi k_B R)^{-1}$ for an Einstein universe, and $T=(8\pi k_B GM)^{-1}$ for a black hole].

2. A modified phase-space numerator with an additional term proportional to s^2.

It has also been shown (Bernard, 1974; Dolan and Jackiw, 1974; Weinberg, 1974) that a quantum field theory at a finite temperature T in Minkowski spacetime can be written as a Euclidean 4-dimensional theory, the 4-space being bounded by two planes separated by a distance $\beta=(k_B T)^{-1}$ along the *temporal* dimension. These boundaries imply a representation of the Euclidean energy in the form of an infinite numerable sum. This feature suggests that there is a relationship between the confinement of a system and the existence of a thermal-like representation which could clarify the *thermodynamic character* of gravitational and cosmological systems.

It is shown in Section 2 that, if under certain conditions a quantity A can be represented by an infinite numerable series of discrete terms, then it can be written as the integral of a thermal function. The thermal representation shows a modified phase-space numerator if the first n terms of the series are absent. In Section 3 some examples, such as the zero-point energy density between two parallel plates or in an Einstein universe at zero temperature, are discussed. Bounded systems at nonvanishing temperature (and therefore with a double confinement) are discussed in Section 4.

In Section 5 some comments on accelerated or gravitational systems are made before the paper closes with a few final remarks in Section 6. Henceforth, the units used are such that $\hbar=1=c$.

2. DISCRETE SUMS, CONFINEMENT, AND THERMAL REPRESENTATIONS

Let A be a conserved quantity which can be represented as an infinite numerable series of discrete terms. In particular, assume that A may be written as

$$A=\sum_{n=0}^{\infty} ng(n^2) \tag{2.1}$$

with $g(0)$ a finite constant.

Two examples of this representation are (see below): the zero-point energy E_0 of a free massless scalar field inside a one-dimensional box $(E_0 \sim \sum_{n=0}^{\infty} n)$, and the zero-point energy of the free electromagnetic field between two parallel plates $[E_0 \sim \sum_{n=0}^{\infty} n^3$ (Casimir, 1948)].

If $g(n^2)$ is expanded as a power series

$$g(n^2) = \sum_{r=0}^{\infty} a_r n^{2r} \tag{2.2}$$

and substituted in equation (2.1), one gets, after interchanging the sums over n and r, that

$$A = \sum_{r=0}^{\infty} a_r \sum_{n=0}^{\infty} n^{2r+1} = \frac{1}{2} \sum_{r=0}^{\infty} a_r \sum_{n=-\infty}^{\infty} |n|^{2r+1} \tag{2.3}$$

Using the Poisson formula (Gelfand and Shilov, 1964),

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} f(\phi)\, e^{2\pi i k\phi}\, d\phi \tag{2.4}$$

one may write the second sum in equation (2.3) in the following form:

$$\frac{1}{2} \sum_{n=-\infty}^{\infty} |n|^{2r+1} = \frac{1}{2} \int_{-\infty}^{\infty} |\phi|^{2r+1}\, d\phi + \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} |\phi|^{2r+1}\, e^{2\pi i k\phi}\, d\phi \tag{2.5}$$

Inserting now the explicit expression for the Fourier transform of the distribution $|\phi|^{2r+1}$ and the definition of the Riemann ζ function $\zeta(m) = \sum_{k=1}^{\infty} k^{-m}$, it follows that

$$\frac{1}{2} \sum_{n=-\infty}^{\infty} |n|^{2r+1} = \frac{1}{2} \int_{-\infty}^{\infty} |\phi|^{2r+1}\, d\phi - 2(-1)^r \frac{\Gamma(2r+2)}{(2\pi)^{2r+2}} \zeta(2r+2) \tag{2.6}$$

On the other hand,

$$\int_0^{\infty} \frac{\omega^{2r+1}}{e^{2\pi\omega} - 1}\, d\omega = \frac{\Gamma(2r+2)}{(2\pi)^{2r+2}} \zeta(2r+2) \tag{2.7}$$

and therefore

$$\frac{1}{2} \sum_{n=-\infty}^{\infty} |n|^{2r+1} = \frac{1}{2} \int_{-\infty}^{\infty} |\phi|^{2r+1}\, d\phi - 2(-1)^r \int_0^{\infty} \frac{\omega^{2r+1}\, d\omega}{e^{2\pi\omega} - 1} \tag{2.8}$$

Inserting equation (2.8) in equation (2.3) and using equation (2.2), one gets

$$A = \frac{1}{2} \int_{-\infty}^{\infty} |\phi| g(\phi^2)\, d\phi - 2 \int_0^{\infty} \frac{\omega g(-\omega^2)}{e^{2\pi\omega} - 1}\, d\omega \tag{2.9}$$

The second term looks like a Planckian with a *temperature* related to ω and a phase-space density modified by a factor $g(-\omega^2)/\omega^2$.

Let us now consider a conserved quantity A that can be written as an infinite sum over half-integers; explicitly,

$$A = \sum_{n=1/2,3/2,\ldots}^{\infty} ng(n^2) \tag{2.10}$$

Rewriting this as

$$A = \sum_{n=0,1,\ldots}^{\infty} \left[\frac{n}{2} g\left(\frac{n}{4}\right)^2 - ng(n^2)\right] \tag{2.11}$$

and using equations (2.2), (2.4), and (2.7), one obtains

$$A = \sum_{r=0}^{\infty} \tfrac{1}{2}(-1)^r a_r[1-2^{-(2r+1)}] \int_{-\infty}^{\infty} |\phi|^{2r+1}\, d\phi + 2\int_0^{\infty} \frac{\omega g(-\omega^2)\, d\omega}{e^{2\pi\omega}+1} \tag{2.12}$$

In this case the second term shows a Fermi–Dirac thermal denominator with a *temperature* that is again related to ω. Also, the factor $g(-\omega^2)/\omega^2$ modifying the phase-space density is present.

3. EXAMPLES

Let us now consider some examples.

(i) *A Free Massless Scalar Field ϕ Inside a One-Dimensional Box of Length L*

Imposing the periodic boundary condition

$$\phi(x=L) = \phi(x=0) \tag{3.1}$$

and defining $k \equiv 2\pi\omega/L$, we obtain the zero-point energy E_0 given by

$$\begin{aligned} E_0 = \frac{\pi}{L}\sum_{n=0}^{\infty} n &= \frac{\pi}{L}\left[\int_0^{\infty} \phi\, d\phi - \frac{2}{(2\pi)^2}\zeta(2)\right] \\ &= \frac{\pi L}{(2\pi)^2}\left[\int_0^{\infty} k\, dk - 2\int_0^{\infty} \frac{k\, dk}{e^{kL}-1}\right] \end{aligned} \tag{3.2}$$

For the zero-point energy density ρ_0 one has

$$\rho_0 \equiv \frac{E_0}{L} = \frac{1}{4\pi}\left[\int_0^{\infty} k\, dk - 2\int_0^{\infty} \frac{k\, dk}{e^{kL}-1}\right] \tag{3.3}$$

The second term corresponds to a thermal representation of the negative Casimir potential energy.

(ii) Electromagnetic Field between Two Metallic Plates Separated by a Distance L

This problem was originally discussed by Casimir (1948). The field vanishes at the surface of both plates, implying untwisted boundary conditions (Birrel and Davies, 1982), the zero-point energy E_0 is now proportional to $-\sum_{n=0}^{\infty} n^3$, which gives a zero-point energy density ρ_0 in terms of the integrals

$$\int_0^\infty k^3\,dk - 2\int_0^\infty \frac{k^3\,dk}{e^{kL}-1} \tag{3.4}$$

As in the one-dimensional problem, the last term is a thermal representation of the negative Casimir potential energy.

(iii) Electromagnetic Field between a Conducting Plate and a Permeable One

This problem has been discussed by Boyer (1974). The field vanishes at the conducting plate, but is maximal at the permeable plate. Thus, one has twisted boundary conditions (Birrel and Davies, 1982). Subtracting a physically unobservable divergent term, it is found that the zero-point energy density ρ_0 is given by

$$\rho_0 = 2\int_0^\infty \frac{\omega^3\,d\omega}{e^{\omega}+1} \tag{3.5}$$

Note that the Fermi–Dirac denominator is related to the twisted boundary conditions.

(iv) Einstein Universes

The zero-temperature energy density ρ_0 of a free massless field with spin s in an Einstein universe of radius R has been found to be given as follows.

(a) $s=0$. Ford (1975) obtains

$$\rho_0 = \frac{1}{2\pi^2}\frac{1}{R^3}\sum_{n=0}^{\infty}\frac{n^3}{2R} = \frac{1}{4\pi^2}\left[\int_0^\infty k^3\,dk + \int_0^\infty \frac{k^3\,dk}{e^{2\pi kR}-1}\right] \tag{3.6}$$

The potential energy is positive in this case.

(b) $s=1/2$. For (1976) obtains

$$\rho_0 = \frac{1}{2\pi^2}\frac{1}{R^3}(2)\sum_{n=1/2,3/2,\ldots}^{\infty}\frac{1}{2R}n\left(n^2-\tfrac{1}{4}\right) \tag{3.7}$$

which leads to a physically observable value given by

$$\int_0^\infty \frac{k(k^2+1/4R^2)}{e^{kR}+1}\,dk \tag{3.8}$$

The factor 2 before the summation sign comes from the two helicity states when the spin $s\neq 0$. The modified phase-space density $k^2\to k^2+(4R^2)^{-1}$ is related to the absence of the $n=1/2$ contribution in equation (3.7).

(c) $s=1$. In this case Ford (1976) obtains

$$\rho_0=\frac{1}{2\pi^2R^3}(2)\sum_{n=0}^{\infty}\frac{1}{2R}n(n^2-1) \tag{3.9}$$

As in equation (3.7), the factor 2 before the summation sign comes from the two helicity states when the spin $s\neq 0$. The $n^2-1=(n+1)(n-1)$ degeneration shows the absence of the $n=1$ long-wave contribution due to the periodicity conditions in the 3-sphere (in a similar way, the $n_1=1$, $n_2=n_3=0$ modes are absent for the electromagnetic field inside a metallic cubic cavity of side L).

From equation (3.9) one may write

$$\rho_0=\frac{1}{2\pi^2}\left[\int_0^\infty k(k^2+R^{-2})\,dk+\int_0^\infty \frac{k(k^2+R^{-2})\,dk}{e^{kR}-1}\right] \tag{3.10}$$

Recall once again that the modification in the phase-space density $k^2\to k^2+R^{-2}$ is due to the n^2-1 degeneration in equation (3.9), which in turn is due to the absence of the $n=1$ long-wave contribution.

(d) Higher Spin Values. As far as we know, the problem has not been solved in a consistent way for $s\geq 3/2$. One might assume that the previous process is valid for arbitrary values of the spin s (either integer or half-integer). However, such a conjecture has not been proved, and when explicitly writing the equation corresponding to an n-dimensional space one gets the n^2-s^2 degeneration with an integer (half-integer) value of n for integer (half-integer) spin s. For $s\geq 3/2$ there is a negative contribution from the n values in the interval $0<n<s$, a feature that is rather unpleasant.

Conserved tensors for arbitrary spin s, which are nonnegative in n-dimensional space, have been considered by Hacyan (1985). While his results only refer to uniformly accelerated frames, they suggest that the correct phase-space numerator must be of the form

$$[\omega^2+s^2a^2][\omega^2+(s-1)^2a^2]\ldots$$

the last term of the product being ω^2 or $\omega^2+(a/2)^2$, according to whether the spin is an integer or a half-integer number. This is consistent with our previous discussion: the phase-space density is modified because all the modes with wave number $n \leq s$ are eliminated.

The preceding analysis and examples show that the presence of a representation with a *thermal* denominator is related to the discrete character of the energy sum (which in turn is due to the confined character of the system). For untwisted boundary conditions the sum is carried out over integer numbers and one finds a Planckian denominator, while for twisted boundary conditions the sum is over half-integer numbers and the denominator is a Fermi–Dirac distribution. The presence of a *modified phase-space* numerator is related (in a spin-dependent way) to the absence of long-wave modes, and the absence of long-wave modes is due to the boundary conditions.

4. BOUNDED SYSTEMS AT A NONVANISHING TEMPERATURE

As previously stated, Bernard (1974), Dolan and Jackiw (1974), and Weinberg (1974) have shown that the partition function for a quantum field theory in a Minkowskian spacetime at an inverse temperature β is equivalent to that for an Euclidean 4-dimensional field theory with the fourth dimension, or Euclidean time τ, bounded by two parallel planes a distance β apart. Due to this confinement, the energy in the Euclidean case shows discrete eigenvalues, and, as in the examples of Section 3, the thermal representation (canonical ensemble) is related to the confinement of the system along the Euclidean temporal direction.

One may now consider a spatially confined system at a nonvanishing temperature T. There will be a double discreteness, one associated with the spatial confinement and one related to the Euclidean temporal confinement (which in turn is due to the nonnull temperature).

As an example, let us consider a free, massless scalar field ϕ inside a one-dimensional box of length L at a finite inverse temperature β. Imposing untwisted periodic boundary conditions for both the x and τ directions $\phi(x, \tau=\beta)=\phi(x, \tau=0)$ and $\phi(x=L, \tau)=\phi(x=0, \tau)$, one obtains the zero-point energy $E_0(\beta)$ as

$$E_0(\beta)=\frac{1}{2}\frac{2\pi}{L}\sum_{n=0}^{\infty} n \coth\left(\frac{n\pi\beta}{L}\right)$$

$$=\frac{\pi}{L}\int_0^{\infty} \phi \coth\left(\frac{\pi\beta\phi}{L}\right) d\phi+\frac{\pi}{2L}\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}|\phi|$$

$$\times \coth\left(\frac{\pi\beta|\phi|}{L}\right) e^{2\pi i k\phi}\, d\phi \tag{4.1}$$

and the zero-point energy density $\rho_0(\beta)$ as

$$\rho_0(\beta) \equiv \frac{E_0}{L} = \frac{1}{4\pi^2} \int_0^\infty k \coth\left(\frac{k\beta}{2}\right) dk$$
$$= \frac{1}{4\pi} \int_0^\infty k\, dk + \rho_\beta + \rho_L + \rho_{\beta L} \tag{4.2}$$

where

$$\rho_\beta \equiv \frac{1}{2\pi} \int^\infty \frac{k\, dk}{e^{k\beta} - 1} \tag{4.3a}$$

$$\rho_L \equiv -\frac{1}{2\pi} \int_0^\infty \frac{k\, dk}{e^{kL} - 1} \tag{4.3b}$$

$$\rho_{\beta L} \equiv \frac{2}{\pi} \sum_{r=1}^\infty \sum_{n=1}^\infty \frac{(n/L)^2 - (r/\beta)^2}{(n^2\beta/L - r^2 L/\beta)^2} \tag{4.3c}$$

Note that ρ_L is again the negative Casimir potential energy for $T = 0$ [equation (3.3)].

The "interference term" $\rho_{\beta L}$ is equal to zero if $\beta = L$, and $\rho_{\beta L} \to 0$ if $\beta \gg L$ or $L \gg \beta$. Therefore, if $\beta \gg L$, then

$$\rho_0(\beta) \to \frac{1}{4\pi} \int_0^\infty k\, dk + \rho_\beta \tag{4.4a}$$

and if $\beta \ll L$, then

$$\rho_0(\beta) \to \frac{1}{4\pi} \int_0^\infty k\, dk + \rho_L \tag{4.4b}$$

The analogous structure of ρ_β and ρ_L is associated with the confinement in the τ and x Euclidean directions, respectively.

A second example is the electromagnetic field in an Einstein universe of radius R at an inverse temperature β; one then has

$$\rho_0(\beta) = \frac{1}{2\pi^2 R^3} \sum_{n=0}^\infty \frac{1}{R} n(n^2 - 1) \coth\left(\frac{n\beta}{R}\right) \tag{4.5}$$

i.e., a linear combination of equation (4.2) and

$$\frac{1}{4\pi^2 R^4} \sum_{n=0}^\infty n^3 \coth\left(\frac{n\beta}{R}\right)$$
$$= \frac{1}{4\pi^2} \left(\int_0^\infty \frac{k^3\, dk}{e^{kR} - 1} + 2 \int_0^\infty \frac{k^3\, dk}{e^{2\beta k} - 1} \right) + \frac{1}{\pi\beta^4} \sum_{n=1}^\infty \sum_{r=1}^\infty$$
$$\times \left[\frac{9}{[n^2 + (r\pi R/\beta)^2]^2} - \frac{32 n^2}{[n^2 + (r\pi R/\beta)^2]^3} + \frac{24 n^4}{[n^2 + (r\pi R/\beta)^2]^4} \right] \tag{4.6}$$

that is, $\rho_0(\beta)$ is the sum of the thermal and Casimir terms (both positive) and an "interference term."

5. NONINERTIAL SYSTEMS

The "thermal character" of the radiation in gravitational or accelerated systems has been investigated in the work mentioned at the beginning of Section 1. The expectation value of the energy-momentum tensor and the energy absorption rate of a uniformly accelerated detector can be given a thermal representation with a temperature proportional to the inverse of the mass of the gravitational field source or proportional to the acceleration of the system.

Near the horizon of the Schwarzschild black hole of mass M, Sciama *et al.* (1981) found that

$$\langle B|T_\mu^\nu|B\rangle_r \xrightarrow[R\to 2M]{} -\frac{h(s)}{2\pi^2(1-2M/R)^2} \times \int_0^\infty \frac{\omega(\omega^2+\chi^2 s^2)\,d\omega}{e^{2\pi\omega/\chi}-(-1)^{2s}}\,\mathrm{diag}(-1,\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}) \tag{5.1}$$

where $\chi=1/4M$, $\langle B|$ is the Boulware vacuum, the index r stands for renormalized, and s is the spin of the massless free field ($s=0$, 1/2, 1).

Candelas and Deutsch (1977, 1978) found that, far from an accelerated mirror, the energy-momentum tensor of the field is given by

$$\langle T_\mu^\nu\rangle_r = -\frac{h(s)}{2\pi^2}\int_0^\infty \frac{\omega(\omega^2+a^2s^2)\,d\omega}{e^{2\pi\omega/a}-(-1)^{2s}}\,\mathrm{diag}(-1,\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}) \tag{5.2}$$

The energy density is negative in both equations (5.1) and (5.2). On the other hand, Hacyan *et al.* (1985) found a thermal representation for the expectation value of the energy density as seen in an accelerated system without a mirror, and the energy density turned out to be positive.

The energy absorption rate of an accelerated detector immersed in a scalar field ϕ is given by (Sciama *et al.*, 1981)

$$\Pi(\omega)=\int_{-\infty}^{\infty} e^{-i\omega t}\langle\hat{\phi}(t)\hat{\phi}(0)\rangle\,dt=\frac{1}{a\pi}\frac{\omega}{e^{2\pi\omega/a}-1} \tag{5.3}$$

where a is the constant proper acceleration, and again a thermal denominator is present.

The above formulas are very similar to the ones found for an Einstein universe, with the inverse of the acceleration replacing the radius R (Candelas and Dowker, 1979). This similarity between the results for Einstein and Rindler universes suggests that again there must be boundaries for the Rindler coordinates. As was pointed out in Sections 1 and 4, a Minkowskian field theory at a finite temperature is equivalent to a Euclidean theory ($t \to i\tau$) with $0 \leq \tau \leq \beta$. The confinement in the τ direction implies a discrete spectrum and the partition function of the canonical ensemble.

The Rindler coordinates for an accelerated particle along the x axis are given by

$$x = a^{-1} \cosh(a\sigma) \qquad \text{and} \qquad t = a^{-1} \sinh(a\sigma) \tag{5.4}$$

where a is the acceleration and σ is the proper time in the accelerated system. If, as in the Minkowskian thermal case, one moves to the equivalent Euclidean space by means of the substitution $\sigma \to i\eta$, equations (5.4) become

$$x = a^{-1} \cos(a\eta) \qquad \text{and} \qquad \tau = a^{-1} \sin(a\eta) \tag{5.5}$$

where η is the Euclidean proper time and $\tau = it$. The hyperbolas become circles and one then has the periodicity condition $\phi(\eta = 2\pi/a) = \phi(\eta = 0)$. This periodicity implies a discrete spectrum and a thermal representation.

Note that equations (5.5) imply a confinement for both the Euclidean time τ and the x direction. This double confinement for Euclidean coordinates in the laboratory system is similar to the one found in Section 4 for a spatially confined system at a finite temperature. In the laboratory system one may therefore think of a "mixture" of thermal and potential energy, although in the accelerated system one may talk of a Euclidean "energy" associated with rotations by the "angular" proper time. A detailed analysis of this point and of equations (5.1)–(5.3) along these lines is currently under progress.

6. FINAL REMARKS

It has been shown that if a quantity A can be written as an infinite sum of numerable terms (over integer or half-integer values), then the quantity A can also be represented as an integral "thermal" spectrum with a Planckian or a Fermi–Dirac denominator and a modified phase space. The thermal denominator is associated with the discreteness of the n sum and the modified effective phase space with the absence of long-wave contributions (small n).

In the examples considered, the discreteness was due to the confinement in the configuration space (one-dimensional box, two parallel plates, and Einstein universes of radius R) or in the temporal dimension τ ($t \to i\tau$) in a 4-dimensional Euclidean space (thermal system). For $s > 0$, the absence of long-wave modes (due to the boundary or periodicity conditions) is responsible for the modification of the phase space.

Although more work is needed, the discussion presented in this paper suggests that the confinement of the system is the main ingredient in the thermodynamic representation of gravitational or cosmological problems. This confinement would imply "absolute ignorance" about the region outside it and the thermodynamic features would therefore follow.

It is also worth remarking that, although one may define a "temperature" from the exponential term of the Planckian or Fermi–Dirac denominator, neither $\hbar$ nor k_B is present in the exponential. The thermal denominator and the modified phase-space numerator depend on the presence of waves with angular momentum in a bounded spacetime. The quantum physics appears only through $\hbar$ as a multiplicative factor in T_μ^ν coming from a normalization associated with the zero-point energy for each mode $\hbar\omega/2$. Strictly speaking, one could say that the problem does not have a thermodynamic character nor a quantum field one.

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